Section 3.2 Matrices & Gaussian Elimination

Matrix : array of numbers, with rows and columns

$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ (2×2) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 2 & 1 \end{bmatrix}$ (2×3)

(rows × cols)

"matrix is 2×2"   "matrix is 3×3"

a column vector is $m\times 1$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

row vector is $1\times n$ $[x_1 \;\; x_2 \;\; x_3]$

★ tuples like (x_1, x_2, x_3) same as column vectors $\left(\text{ex: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$

- Matrices provide shorthand for solving systems of linear eqs :

★ optional

$$\begin{cases} x + 3y = 9 & (R_1) \\ 2x + y = 8 & (R_2) \end{cases} \longleftrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \text{right side} \end{array} \\ \left[\begin{array}{cc|c} 1 & 3 & 9 \\ 2 & 1 & 8 \end{array}\right] \end{array}$$

$$(R_2 \to R_2 - 2R_1) \begin{cases} x + 3y = 9 \\ 0 - 5y = -10 \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 3 & 9 \\ 0 & -5 & -10 \end{array}\right]$$

x "eliminated"

$$\begin{array}{c} \begin{array}{cccc} x & y & = & \# \end{array} \\ \left[\begin{array}{cc|c} 1 & 3 & 9 \\ 0 & -5 & -10 \end{array}\right] \end{array} \longleftrightarrow \begin{cases} x + 3y = 9 & (R_1) \\ \quad -5y = -10 & (R_2) \end{cases}$$

$(R_2) \Longrightarrow \underline{y = 2}$

Now "back-substitute" $y=2$ into (R_1):

$x = 9 - 3y = 9-6 = 3$ "bottom $\to$ top"

$\boxed{x=3}$

$(x,y) = (3,2)$ is the unique solution

solution vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

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first nonzero entry in row is a "pivot"

$$\left[\begin{array}{cc|c} \boxed{1} & 3 & 9 \\ 0 & \boxed{-5} & -10 \end{array}\right]$$

Matrix is in (row-) echelon form (EF):

1) Numbers below a pivot are all 0

2) Any rows of all 0 moved to bottom.

Ex

$$\left\{\begin{array}{l} 3x - 8y + 10z = 22 \\ x - 3y + 2z = 5 \\ 2x - 9y - 8z = -11 \end{array}\right\} \quad 1$$

$$\left[\begin{array}{ccc|c} 3 & -8 & 10 & 22 \\ 1 & -3 & 2 & 5 \\ 2 & -9 & -8 & -11 \end{array}\right]$$

"coefficient matrix" (for system 1)

"augmented matrix" (for system 1)

$(R_2 \leftrightarrow R_1)$ $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 3 & -8 & 10 & 22 \\ 2 & -9 & -8 & -11 \end{array}\right]$

(swap/scale/add are "elementary row operations")

$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$ $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & -3 & -12 & -21 \end{array}\right]$

"x_1 eliminated from R_2, R_3." Now eliminate x_2 from R_3

$(R_3 \to R_3 + 3R_2)$ $\begin{array}{c} \left[\begin{array}{ccc|c} \boxed{1} & -3 & 2 & 5 \\ 0 & \boxed{1} & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$ (EF)

pivot rows

pivot columns

x_1, x_2 are pivot variables

x_3 is a free variable (no pivot in column)

Let $x_3 = t$ (free param)

Change back $\begin{cases} x_1 - 3x_2 + 2x_3 = 5 \\ \quad\; x_2 + 4x_3 = 7 \\ \quad\quad\quad\; 0 = 0 \end{cases}$

$R_2 \Rightarrow$ $x_2 = 7 - 4x_3 = 7 - 4t$

$R_1 \Rightarrow$ $x_1 = 5 + 3x_2 - 2x_3$

$= 5 + 3(7 - 4t) - 2t$

$x_1 = 26 - 14t$

Solution vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 26 - 14t \\ 7 - 4t \\ t \end{bmatrix} = \begin{bmatrix} 26 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14 \\ -4 \\ 1 \end{bmatrix}$$

This method of solving by <u>row reducing</u> to echelon form is the "<u>method of (Gaussian) elimination</u>." Works in more variables too.

<u>Ex</u>

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 8x_2 + 7x_3 = 20 \\ 2x_1 + 7x_2 + 9x_3 = 23 \end{cases}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\left(R_2 \to \tfrac{1}{2}R_2\right) \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\left(R_3 \to R_3 - 3R_2\right) \quad \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \end{array} \quad \text{(EF)}$$

change back

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 + 2x_3 = 4 \\ \underline{x_3 = 3} \end{cases}$$

$\underline{x_2} = 4 - 2x_3 = \underline{-2}$

$\underline{x_1} = 4 - 2x_2 - x_3 = 4 + 4 - 3 = 5$

<u>solution vector</u> $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

Method of Elimination

(Basically the standard way you learned previously)

Idea (with 2 eqs/rows $\begin{cases} R_1, \\ R_2 \end{cases}$ 2 vars x, y

0) (Free to swap/scale rows if desired.)

1) Set $(R_2 \to R_2 + cR_1)$, where c is chosen to eliminate x from R_2 *

2) Assuming bottom row (R_2 here) isn't a contradictory eq. ($0=1$, etc.), solve* for y.

3) Use $y = \underline{\dots}$ value an "back-substitute" value into R_1, then solve for x.

* if last row is $0=0$, then let t be a free parameter, set $y=t$, and substitute that into R_1, then solve for x (in terms of t)

* If more than 2 rows, eliminate x from all rows under R_1 at step 1) and then continue.

Method of Elimination + back substitution (3 vars)

(3 rows in example)

0) (Swap/Scale rows if desired)

1) Use R_1 to eliminate x from R_2 and R_3 (and others if more)

2) Use (new) R_2 to eliminate